Confined chaotic behavior in collective motion for populations of globally coupled chaotic elements

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The Lyapunov exponent for collective motion is defined in order to characterize chaotic properties of collective motion for large populations of chaotic elements. Numerical computations for this quantity suggest that such collective motion is always chaotic, whenever it appears. Chaotic behavior of collective motion is found to be confined within a small scale, whose size is estimated using the value of the Lyapunov exponent. Finally, we conjecture why the collective motion appears low dimensional despite the actual high dimensionality of the dynamics. [S1063-651X(99)06802-6]

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I. INTRODUCTION

Systems consisting of large populations of interacting dynamical elements are widely distributed in nature, from communities of ants to biological cell assemblies and neural networks. Many such populations are similar in the respect that they exhibit various kinds of collective behavior. One possible approach to the mathematical study of such collective behavior is to concentrate on certain idealized models such as interacting limit cycles and chaotic maps. An abundance of literature has been devoted to the study of these models [1–6]. It was suggested in some foregoing studies [6–8] that a population of this type as a whole exhibits low-dimensional behavior. This seems to be true even when the individual elements appear to be mutually uncorrelated, a situation in which the population could only be regarded as a dynamical system of extremely high dimension. It still remains unclear whether such collective behavior could be understood in terms of low-dimensional dynamical systems.

In globally coupled tent maps, collective motion has been understood partly from a macroscopic viewpoint [9,10]. In these studies, the collective motion was discussed only with regard to its size, without consideration of its detailed structure. Some beautiful relations regarding the macroscopic properties of the collective motion were explored and a phase diagram was proposed. These results, however, do not represent progress toward an understanding of collective motion in terms of low-dimensional dynamical systems. In this paper, we first focus on the dynamical properties of collective motion. We define the Lyapunov exponent characterizing this collective motion and then a special relation involving this quantity is found. This relation suggests that the collective motion is always chaotic, whenever it appears. Furthermore, chaotic behavior in the collective motion is found to be confined within a small scale whose size is re-

lated to the value of the Lyapunov exponent. Finally, on the basis of our numerical results, we conjecture an answer to the following question: Why does the collective motion appear low dimensional despite the actual high dimensionality of the dynamics?

II. MODEL: GLOBALLY COUPLED TENT MAPS

As model systems, we have chosen globally coupled tent maps, because these systems are particularly well suited for detailed numerical analysis: We can deal with an ideal limit of infinitely large population.

Globally coupled maps (GCM) are given by an assembly of N elements whose behavior is determined by N identical maps with all-to-all coupling. The individual elements are then under the influence of a common internal field which may be referred to as a mean field [5]. We assume that a single isolated element evolves according to $X_{n+1}=f(X_n)$, where n designates discrete time steps. Under the interaction through the mean field h_n, the ith element is then assumed to evolve as

$$X_{n+1}^{(i)}=f(X_n^{(i)})+Kh_n, \qquad (1)$$

where K is the coupling strength. In this paper, we consider the situation in which f is a tent map:

$$f(X)=-a|X|+\frac{a-1}{2}, \qquad (2)$$

and the mean field h_n is defined as

$$h_n\equiv\frac{1}{N}\sum_{j=1}^{N}f(X_n^{(j)}). \qquad (3)$$

Thus, our system is characterized by the two parameters a and K in addition to the total number of elements N. Each tent map has a band-splitting point $a=\sqrt{2}$ and we thus stipulate that a satisfies $\sqrt{2}<a<2$. It is thereby ensured that the population will never split into subpopulations. If the system size N is finite, this finiteness becomes the source of fluctuations of the mean field. Such an effect may obscure pure

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collective motion. Thus we work with the limit of large N. In fact, we confirmed that finite size effects can be regarded as noise acting on the pure collective motion. For $N \to \infty$, the population dynamics of GCM can be described by the Frobenius-Perron equation [11] for the distribution $\rho(X)$:

$$\rho_{n+1}(X) = \int \delta[X - f(X') - Kh_n]\rho_n(X')dX', \qquad (4)$$

$$h_n = \int f(X')\rho_n(X')dX'. \qquad (5)$$

We worked out a numerical scheme for the exact integration of Eqs. (4) and (5) whose precision is limited only by round-off errors [12]. Numerical results in this paper were obtained using only a single initial distribution, uniform over the interval $[f \circ f(0):f(0)]$.

III. BRIEF REVIEW OF PREVIOUS STUDIES

We review here several results for collective motion in globally coupled tent maps. Previously, this system was not reported to show nontrivial behavior of h_n in contrast with globally coupled logistic maps. Ershov and Potapov [13] worked analytically on this system [with Eqs. (4) and (5)], and suggested that the size of the fluctuations of h_n is scaled as $\exp(-K^{-2})$. Contrary to the previous numerical suggestions, their conclusion implied that the system would exhibit nontrivial behavior even in vanishingly small values of K. In order to confirm it, more precise calculation was required. Morita [12] reported a good numerical scheme for integration of Eqs. (4) and (5). Especially in the case of the globally coupled tent maps, the integration becomes exact, and then precision is limited only by round-off errors. Using a similar scheme, Nakagawa and Komatsu [9] found that the size of the collective motion is scaled as $\exp(-K^{-1})$ at special parameter values of a which they called "golden values." Although the scaling form is different from that of Ershov and Potapov, it also implies that the collective motion exists even in vanishingly small values of K. They also reported that a phase diagram for the collective motion resembles that of the Arnold tongues, where each tongue is connected with the golden values of a. Another analytical approach to the collective motion was introduced by Chawanya and Morita [10]. Through the investigation of the stability for the fixed point of h_n, their theory shed light on the bifurcation of the collective motion. Their results were almost consistent with the numerical conclusion in [9]. Thus collective behavior for the globally coupled tent maps is now fairly explored. However, the dynamical nature of the collective motion is still open, and this *is* just the theme of present paper.

IV. GROWTH OF COMPLEXITY IN COLLECTIVE MOTION

Collective motion can be observed by studying the dynamics of the order parameter h_n given by Eq. (5). Figures 1(a) and 1(b) are return maps for this order parameter. There we see that the fluctuations of h_n undergo quasiperiodic motion for small K, but for larger K they display more complicated motion. Roughly speaking, larger values of K generi-
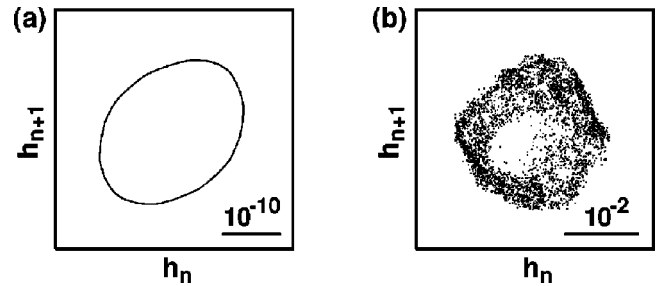

FIG. 1. Two types of collective motion at $a = g_5$ [defined by $-\log_a(2-a) = 4$] for the return map of h_n. (a) For $K = 0.1$, we find a torus representing quasiperiodic motion. (b) For $K = 1.0$, the return map displays more complicated toruslike motion, possibly accompanied by fine structure.

cally result in more complicated collective behavior over all values of a.

In the case of small K, the collective motion appears quasiperiodic. Despite the familiar appearance of this collective motion, we should discuss carefully the nature of these dynamics, because the system has an infinitely high dimension. The chaotic structure on the torus seems to grow gradually, without an indication of bifurcation from quasiperiodic to the chaotic behavior. It may be the case that the structure on the torus persists as long as there is a nonvanishing interaction. In following sections, we focus on the chaotic properties of the collective motion.

V. EXPONENTIAL GROWTH OF DISTURBANCES

In order to characterize the dynamical properties of the collective motion, we wish to define a value similar to the Lyapunov exponent to characterize the collective motion. We note that it is unfeasible to calculate conventional Lyapunov exponents based on the phase-space structure of the system, because our system is of an infinitely high dimension. However, what we wish to concentrate on is not the phase-space structure but the dynamical properties of the collective motion. Hence we attempt to define a quantity similar to the Lyapunov exponent to describe the dynamics of the collective motion.

We first add a small disturbance to the distribution $\rho_n(X)$ at time step n. The mean field h_n is changed to $h_n^{(d)}$ as a result. The difference between the disturbed and nondisturbed mean fields,

$$\delta_l \equiv |h_{n+l}^{(d)} - h_{n+l}|, \qquad (6)$$

depends on l, the time lapsed since the application of the disturbance. Without an interaction ($K = 0$) among the elements in the population, the distribution $\rho_{n+l}^{(d)}(X)$ gradually approaches that of the nondisturbed case $\rho_{n+l}(X)$. This is because the two distributions should realize the same invariant measure of a single tent map. Thus $\delta_l \to 0$ as l becomes large. When $K \neq 0$, collective motion generally occurs. We expect that its dynamical properties can be represented by the behavior of δ_l. If the collective motion is completely quasiperiodic, the difference δ_l will remain on the same order as δ_0, while, if it is chaotic, δ_l will grow exponentially with the increase of l.
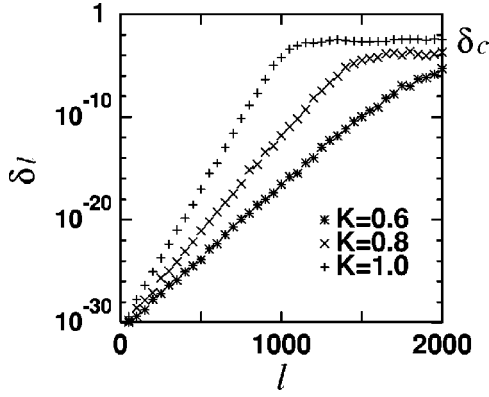
FIG. 2. Exponential growth of the difference δ_l between the order parameters $h_{n+l}^{(d)}$ and h_{n+l}. The vertical scale is logarithmic. Here $a = g_5$ $[-\log_a(2-a) = 4]$, while $K = 0.6$, 0.8, and 1.0. All points represent sample averages of 10 different initial distributions $\rho_n^{(d)}(X)$ with a fixed difference $\delta_0 = 10^{-30}$.

In numerical experiments, δ_l is estimated for one set of parameters (a,K) as follows: The value $|h_{n+l}^{(d)} - h_{n+l}|$ is calculated from various initial distributions of $\rho_n^{(d)}(X)$ with a fixed small difference $\delta_0 \equiv |h_n^{(d)} - h_n|$. The sample average over various $|h_{n+l}^{(d)} - h_{n+l}|$ calculated in this way gives δ_l. In our computations, this average was calculated from either 10 or 100 initial disturbed distributions. We point out here that individual $|h_{n+l}^{(d)} - h_{n+l}|$ actually depends on l in a manner similar to δ_l.

In Fig. 2, we display examples for the exponential growth of the difference δ_l between the mean fields. The initial difference δ_0 is fixed at as small as 10^{-30}. The difference δ_l exhibits a clear dependence on the time step l: δ_l grows exponentially and finally saturates near a certain scale. It is surprising that the exponential growth of δ_l is observed for a considerably large interval of l. It is therefore possible to define the exponent λ which characterizes the exponential growth of δ_l. The exponent λ corresponds to the initial slope in Fig. 2. For values of l at which δ_l displays exponential growth, we can define λ according to

$$\lambda \equiv \frac{1}{l} \ln\left(\frac{\delta_l}{\delta_0}\right). \tag{7}$$

This exponent was found to be independent of the initial disturbance of the distribution for sufficiently small values of δ_0 and sufficiently large values of l. Hereafter, we call the exponent λ the *Lyapunov exponent for the collective motion*.

In Fig. 2 we display three examples of δ_l, where the values of K are 0.6, 0.8, and 1.0 while the value of a is fixed according to $-\log_a(2-a) = 4$. Larger values of K result in larger values of λ. The collective motion is concluded to be of a chaotic nature in every case of Fig. 2. If we were to calculate conventional Lyapunov exponents based on the phase-space structure of this system, a maximal exponent with a value close to $\ln a$, the Lyapunov exponent for a single tent map, would be obtained. We note that the value of λ is generally smaller than $\ln a$. In the case of Fig. 2, $\ln a = 0.48$, which is larger than the values found for λ (for in-
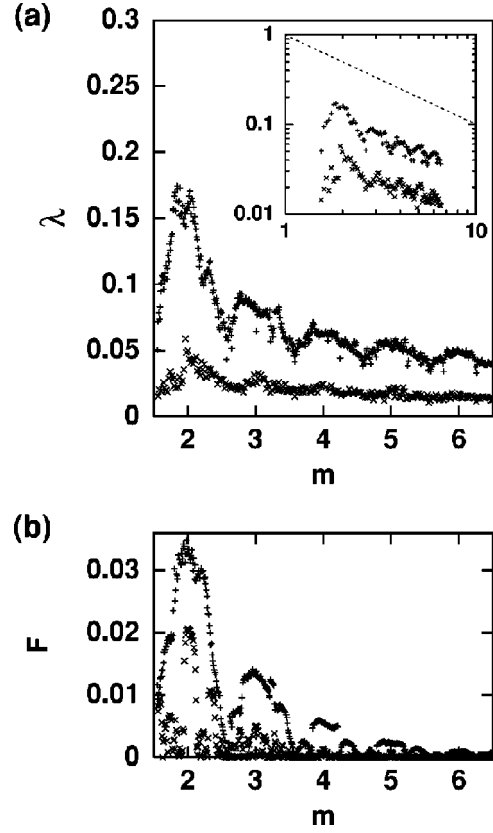

FIG. 3. Hilly structure as a function of m for $K = 1.0$ ($+$) and 0.5 ($\times$). Here $m = -\log_a(2-a)$. (a) m vs Lyapunov exponent λ. In the inset, the same data are displayed on logarithmic scales for both axes. The dotted line indicates $\lambda = m^{-1}$. (b) m vs amplitude F of collective motion. There are several hills and valleys whose locations seem to correspond to those in (a). Some of the golden values of a coincide with integer values of m while some of the silver values coincide with the minimum points of valleys of F.

stance, $\lambda = 0.17$ when $K = 1.0$). This implies that the dynamics of the collective motion have different nature from that of the individual elements.

VI. HILLY STRUCTURE OF THE LYAPUNOV EXPONENT

Figure 3 shows the Lyapunov exponent λ for the collective motion as functions of a [or, more precisely, as functions of $m = -\log_a(2-a)$] at fixed values of K. Here, hilly structure is displayed by the Lyapunov exponent λ, as seen in Fig. 3(a). For comparison, we display the hilly structure corresponding to *macroscopic properties* of the collective motion in Fig. 3(b). These macroscopic properties are represented by the amplitude F of the collective motion,

$$F \equiv \sqrt{\langle (h_n - \langle h_n \rangle)^2 \rangle}. \tag{8}$$

Here $\langle \rangle$ represents a long time average. It is worth noting that the peaks and valleys of these two sets of hilly structure seem to be located at the same values of a. For further detail concerning the hilly structure of F, see [9]. Two special series of parameter values of a, the "golden" and "silver" values (see the following sections for their definition), are the key to understanding the hilly structure. Briefly, the
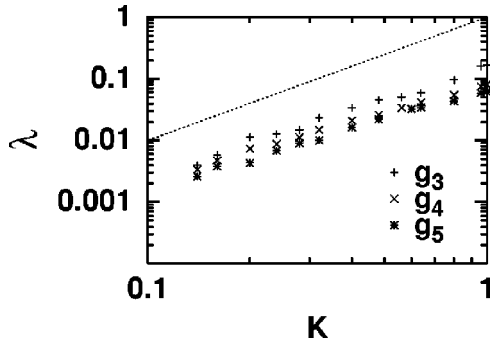
FIG. 4. Scaling relation of the Lyapunov exponent λ for the collective motion with K. Here λ vs K is shown for three golden values g_p, $p=3$–5. $a=g_p$ is equal to $-\log_a (2-a)=p-1$. The dotted line indicates $\lambda=K^2$.

FIG. 5. Decay of the Lyapunov exponent λ for the collective motion near a silver value $s_7=1.81327\ldots$. Here $\Delta a\equiv a-s_7$. $K=0.7$. (a) $-\ln|\Delta a|$ vs λ with logarithmic scales for both axes. The dotted line corresponds to $\lambda\sim(-\ln|\Delta a|)^{-2/3}$. (b) $|\Delta a|$ vs λ with normal scales for both axes. The region where λ decays to 0 is found to be very narrow when λ is plotted in this way.

golden values of the parameter a are situated in the middle of each hill and the silver values are at the minimum points in the valleys. The same situation is found in the case of the Lyapunov exponent λ for the collective motion.

We now concentrate on the differences between these two sets of hilly structure. The hilly structure of the Lyapunov exponent λ exhibits gentle ups and downs over the range of values of a, while the amplitude F displays very sharp ups and downs. The most important difference appears in the valleys of the hilly structure: The value of F decays linearly to 0 in each valley, but the value of λ in each valley seems to remain on the same order as that assumed at the top of the nearest hill. However, we will later show that the values of λ in fact also decay to 0. Decaying will be found to occur only in a very narrow region of a around the minimum point in each valley.

The regions in which the value of λ is close to zero are sufficiently narrow that we can ignore them in estimating the order of the Lyapunov exponent λ as a function of a. For fixed values of K the value of the Lyapunov exponent λ decays as a becomes large. With the definition $m\equiv -\log_a (2-a)$, this decaying tendency is estimated as

$$\lambda\sim m^{-1}, \tag{9}$$

for positive values of K. In Sec. VII, we discuss the exceptional regions of a (i.e., those in which this relationship does not hold).

VII. SCALING RELATION OF THE LYAPUNOV EXPONENT

In this section, we discuss the relation of the exponent λ with the coupling strength K. The value of a is confined to be one of the golden values, while the value of K is changed. Here a golden value, say g_p, is a value of a such that one isolated tent map possesses the property that a trajectory beginning at the peak C of the tent map returns to C after p steps. There exists a sequence of such golden values, parametrized by p. As shown in Fig. 4, it is confirmed that the collective motion becomes increasingly chaotic as the value of K increases. For all the golden values we considered, we have found that the Lyapunov exponent λ grows with the coupling strength K according to the relation $\lambda\sim K^2$. The value of λ also obeys the relation Eq. (9). Thus we find that
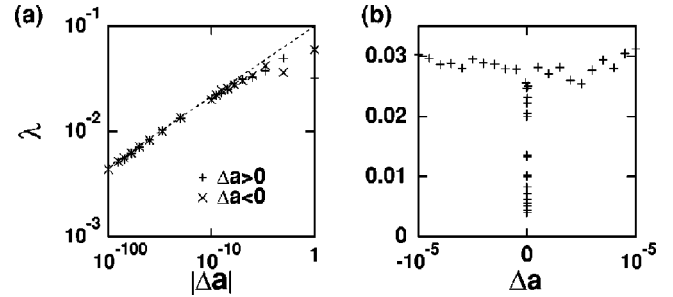
the Lyapunov exponent λ for the collective motion behaves as

$$\lambda\sim\frac{K^2}{m}, \tag{10}$$

for the golden values of a. While we have found this relation by confining our consideration to these golden values, it should be noted that a relation similar to Eq. (10) is expected to hold for all values of a except in very narrow regions, due to the gentleness of the hilly structure of λ. Also note that Eq. (10) in some sense describes the envelope of λ in the (a,K) parameter space, as it does not include the periodic dependence on m or the fine peak structure of λ.

Equation (10) indicates that when there is an interaction between elements (i.e., $K\neq0$) the collective motion is always chaotic. Thus collective motion displaying quasiperiodicity, as is seen in Fig. 1(a), is concluded to actually be chaotic. As the coupling strength K approaches 0 or the parameter a approaches 2 (i.e., $m\to\infty$), the Lyapunov exponent λ for the collective motion vanishes asymptotically.

VIII. LYAPUNOV EXPONENTS AROUND DISAPPEARING POINTS OF THE COLLECTIVE MOTION

The scaling relation for the Lyapunov exponent λ, Eq. (10), leads us to conclude that the collective motion is always chaotic. However, there do exist exceptional cases in which the collective motion disappears. In particular, it disappears at the minimum points in each valley of the hilly structure of the amplitude F. Around these minimum points, the Lyapunov exponent for the collective motion is expected to decay to vanishing or negative values. However, it is difficult to detect this behavior in Fig. 3. We now investigate in detail the nature of the Lyapunov exponent around the minimum points. ''Silver values'' of a are situated at the minimum points in the valleys. The definition of silver values is as follows: When a is identical to s_p, isolated tent maps possess the property that a trajectory beginning at C falls into a fixed point of the tent map after p steps.

Figure 5 shows the decaying behavior of the Lyapunov exponent around a silver value s_p of the parameter a. $|\Delta a|$ represents the distance from a to the silver value s_p. Figure 5 displays a very narrow neighborhood of s_p. The value of λ

becomes small as $|\Delta a|$ approaches 0. As reflected by this figure, the Lyapunov exponent scales with the distance $|\Delta a|$ as

$$\lambda \sim (-\ln|\Delta a|)^{-2/3}. \qquad (11)$$

This relation shows that the Lyapunov exponent decays to 0 as the value of a approaches a silver value. The Lyapunov exponent is nonpositive only at the silver value. There, the collective motion does not appear, so that chaotic nature of this motion is never realized. What we wish to emphasize is that λ falls to zero in only a *very narrow neighborhood* of the silver value [see Fig. 5(b)]. For all other values of a, the Lyapunov exponent is approximately given by Eq. (10). The narrowness of the decaying region around each silver value causes the apparent smoothness of the Lyapunov exponent as a function of a. The motion of the system is chaotic everywhere, except at the minimum point of each valley, where collective motion does not exist. We now conclude that, whenever collective motion appears (i.e., $F \neq 0$), it is chaotic.

IX. MINIMUM STRUCTURE OF THE COLLECTIVE MOTION

Let us now turn in our investigation of the Lyapunov exponent for the collective motion to consideration of the characteristic scale, that is, the scale δ_c at which the exponential growth of δ_l saturates (see Fig. 2). The value of δ_c is determined numerically through the time average of δ_l over an interval of sufficient length just after the saturation. After the saturation of δ_l, it may continue to increase very slowly, until it becomes comparable to the size F of the collective motion. We, however, can choose an appropriate interval for obtaining a valid value of δ_c through the time average of δ_l just after saturation.

Below the scale characterized by δ_c, the collective motion behaves chaotically corresponding to a linear instability of the collective motion. The structure characterized by δ_c is of a scale at which the nonlinearity first appears in the collective motion. The chaotic behavior of the collective motion is confined in this structure. We hereafter call the structure of the collective motion characterized by δ_c the *minimum structure*. We determined an expression for the characteristic scale δ_c of the minimum structure. δ_c was found to depend on the value of K according to a certain scaling relation. As is seen in Fig. 6, for sufficiently small values of K, δ_c behaves as

$$\delta_c \sim \exp\left(-\frac{1}{K^2}\right). \qquad (12)$$

This figure was obtained at a particular golden value of a while the relation (12) was confirmed at every golden value we considered. The relation in Eq. (12) indicates that the minimum structure exists however small the coupling strength K is.

Let us now compare the size δ_c of the minimum structure with that of the collective motion. The amplitude F of the collective motion explored in [9] obeys a different type of scaling relation,
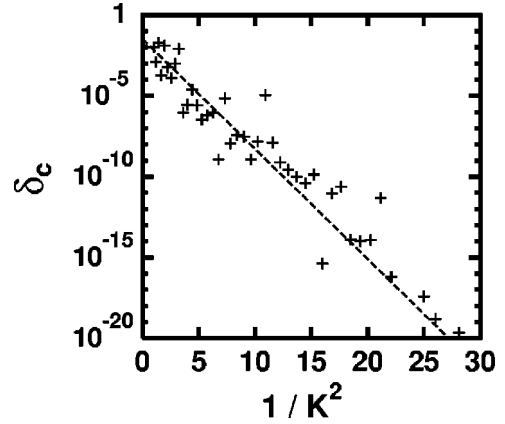

FIG. 6. Scaling relation of the size δ_c of the minimum structure with K. Here δ_c vs $1/K^2$ is shown for the golden value $a = g_4$ $[-\log_a(2-a) = 3]$. The vertical scale is logarithmic. The dotted line indicates $\delta_c \sim \exp(-K^{-2})$.

$$F \sim \exp\left(-\frac{1}{K}\right). \qquad (13)$$

The scaling relations for both δ_c and F are given by monotonically increasing functions of K. Both sizes decay to 0 as K decreases. However, the rate at which this decay occurs is much different for δ_c and F. The size δ_c of the minimum structure becomes much smaller than the amplitude F of the collective motion for sufficiently small values of K, while they are comparable for $K \simeq 1$. The minimum structure and the chaotic behavior within it belong to a collective set of phenomena in the *submacroscopic scale*: They are properties characterizing the smaller scale of the collective motion, while the collective motion is a product of the macroscopic nature of the distribution $\rho_n(X)$.

X. LYAPUNOV EXPONENT AND THE MINIMUM STRUCTURE

For the golden values of the parameter a, the size of the minimum structure is found to be strongly correlated with the value of the Lyapunov exponent λ for the collective motion:

$$\delta_c \sim \exp\left(-\frac{1}{\lambda}\right). \qquad (14)$$

This relation is obtained through the substitution of Eq. (10) into Eq. (12). We conjecture that this relation is not mere coincidence (i.e., its validity is not confined to the golden values) but holds for all sets of (a, K). The characteristic size δ_c corresponding to the chaotic behavior should naturally be related to the Lyapunov exponent. This conjecture has been confirmed through numerical comparison of λ and $-(\log_a \delta_c)^{-1}$. In Fig. 7 we show that these two values depend on a in a very similar manner, although the values of $-(\log_a \delta_c)^{-1}$ are more scattered than those of λ. We thus conclude that the relation

$$\delta_c \simeq a^{-1/2\lambda} \qquad (15)$$
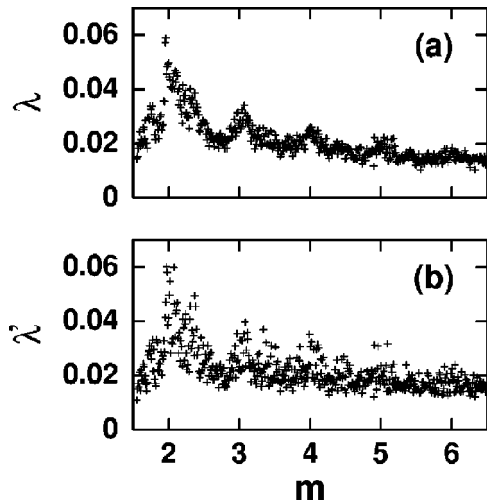
is valid for any set of (a, K).

FIG. 7. Relation between the Lyapunov exponent λ and the size δ_c of the minimum structure. Here $m = -\log_a(2-a)$. This figure is for the case $K = 0.5$. (a) The Lyapunov exponent λ for the collective motion. (b) $\lambda' \equiv (-2\log_a\delta_c)^{-1}$, where δ_c is the size of the minimum structure (also see Fig. 2). We find good correspondence between (a) and (b).

FIG. 8. Locking phenomenon of the collective motion at $a = g_4$ $[-\log_a(2-a) = 3]$ and $K = 0.5$, for the return map of h_n. In the blowups, fluctuations are found around each locked point. The extent of the fluctuations is comparable to the size δ_c of the minimum structure at a corresponding set of parameters (a,K).

The relation in Eq. (15) implies some properties of the minimum structure. As was argued, the Lyapunov exponent λ remains positive except at the silver values of a. Accordingly, the minimum structure exists for almost all sets of (a,K), where it becomes small as the chaotic properties become weak. When the coupling strength K approaches 0, the values of the Lyapunov exponent was found to approach 0 with the relation in Eq. (10). Corresponding to this, the size δ_c of the minimum structure also decreases to 0 with a much larger decreasing rate with K, as was described in Eq. (12). At the silver values of a, the minimum structure does not exist ($\delta_c = 0$), because the collective motion disappears at these points. The value of λ at these points was numerically found to be 0, which is consistent with the nonexistence of the minimum structure ($\delta_c = 0$). The relation in Eq. (15) indicates furthermore that the size δ_c of the minimum structure also exhibits hilly structure as a function of a, like λ and F. In fact, the hilly structure of δ_c has been numerically generated. However, because the values of δ_c found in this study are widely scattered, the functional dependence of δ_c on a was not made completely clear.

It was concluded that the collective motion is almost always chaotic even though it appears quasiperiodic. We now obtain a second conclusion from the relation in Eq. (15) that the chaotic behavior is always confined in a submacroscopic scale given by Eq. (15). If the coupling strength K is sufficiently small, the chaotic behavior is observed only in a much smaller scale than the scale of the collective motion. We argue in a following section that this confinement of the chaotic behavior leads to the low-dimensional appearance of the collective motion.

XI. CHAOTIC FLUCTUATIONS IN THE COLLECTIVE MOTION

The minimum structure is related to a certain observed structure in the collective motion. Consider first the case that
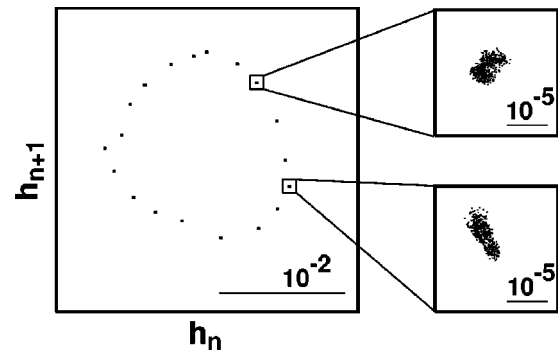
the collective motion locks to some periodic states, as is seen in Fig. 8. Each periodic point possesses a region which is subject to strong fluctuations as is shown in the blowups. The extent of these fluctuations seems to be comparable to the scale δ_c of the minimum structure. When, as in Fig. 1, the collective motion does not lock to periodic states, the structure corresponding to δ_c is similar to the case of the locked periodic states. As is typically observed [see Fig. 1(b)], the collective motion fluctuates around some quasiperiodic motion. The scale of these chaotic fluctuations seems to be on the same order as the value of δ_c.

Accepting the correspondence between the extent of the chaotic fluctuations and the size of the minimum structure, we can interpret the observed collective motion in terms of the macroscopic properties of the collective motion and the minimum structure. If $K \gtrsim 1$, δ_c is comparable to the size F of the collective motion. In this case the collective motion becomes fully developed chaotic motion [see Fig. 1(b)]. When $K \ll 1$, the size δ_c of the minimum structure becomes much smaller than the size F of the collective motion. Here, it is possible for the collective motion to approach a form similar to that displayed by a low-dimensional system, as in the case of the quasiperiodic motion depicted in Fig. 1(a). Our results imply that it is not possible for the system to realize quasiperiodic behavior which lacks chaotic fluctuations. Whenever the collective motion appears, it is always accompanied by chaotic fluctuations. It is important that these chaotic fluctuations are confined within the submacroscopic scale whenever they appear. The collective motion does not realize true quasiperiodic behavior, however it reduces to a form which appears to be quasiperiodic due to the confinement of the chaotic fluctuations within the submacroscopic scale. As a result, the quasiperiodic behavior of the collective motion becomes conspicuous, especially for small values of K, where the characteristic scales of the collective motion and the minimum structure are very different.

We now make a conjecture to answer the question of why the collective motion appears low dimensional despite the actual high dimensionality of the dynamics. Recall now that the collective motion results from dynamics of infinitely high dimension. This is because the present model [Eqs. (4) and (5)] consists of an infinite number of interacting chaotic elements without any mechanism to reduce the number of the
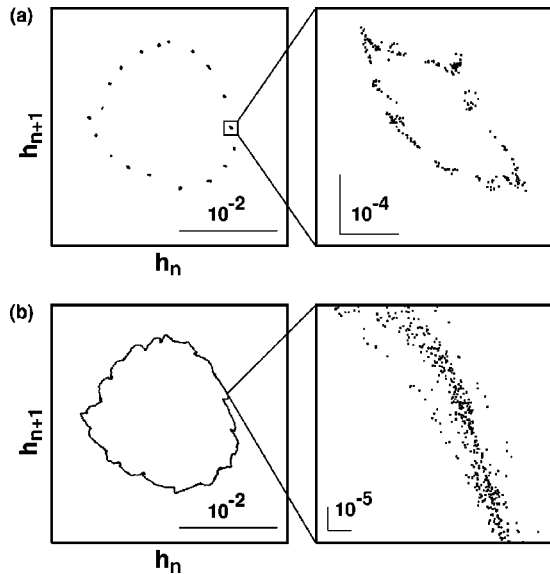
FIG. 9. Examples of collective motion. Here $a = g_4$ $[-\log_a(2-a) = 3]$. Each figure is for the return map of h_n. (a) $K = 0.501$. Each periodic point in Fig. 8 becomes a torus due to the instability of the locked point. For a slightly larger value of K, these small tori exhibit period locking again. (b) $K = 0.507$. The locked state is not stable here. A kind of delicate structure is observed, reflecting the stability of the macroscopic dynamics. Points are seen to be scattered when a pattern is observed at a sufficiently small scale. These fluctuations appear even after a transient behavior has disappeared.

degrees of freedom. It is natural to imagine that the chaotic fluctuations in the submacroscopic scale constitute chaotic behavior of infinitely high dimension. Due to the confinement of the high-dimensional chaos within the submacroscopic scale, the collective motion reduces to a form which appears to be low-dimensional.

XII. MACROSCOPIC BEHAVIOR VERSUS CHAOTIC FLUCTUATIONS

We now use a term *macroscopic behavior* definitely for the coarse-grained behavior of the collective motion below the submacroscopic scale, i.e., below the scale of δ_c of the minimum structure. Macroscopic behavior was observed to exhibit low-dimensional motion such as quasiperiodic and locked periodic motion. We now briefly discuss the variety of the macroscopic behavior for the collective motion. The macroscopic behavior of the collective motion changes very frequently in the parameter space (a,K): typically the quasiperiodic motion repeatedly locks to and releases from some periodic states (see Fig. 8). Examples of the collective motion are displayed in Fig. 9. The locking phenomena of the collective motion are often followed by the instability of the locked points. We find an assembly of tori, as in Fig. 9(a). These tori can also become locked, where each torus in the assembly turns into locked states so that an assembly of the locked periodic states is formed. In such a way, a kind of hierarchical structure of locked states can be realized when the creation of tori, resulting from the destabilization of periodic points, is repeated again and again, leading to the formation of assemblies of tori. When the collective motion does not lock, it comes to possess fine structure reflecting a

delicate dynamical property for the macroscopic behavior [Fig. 9(b)].

As is seen in the above examples, the macroscopic motion can become complicated, in some cases possessing fine structure, such as a kind of hierarchical structure. We here emphasize that, without regard to the dynamical properties of the macroscopic scale, the minimum structure inevitably exists along with the macroscopic behavior of the collective motion. For all patterns of the macroscopic behavior, the chaotic behavior only remains below the submacroscopic scale. In this scale the dynamical properties of the macroscopic behavior are destroyed, whereas the chaotic properties in the submacroscopic scale do not seem to affect the collective behavior in the macroscopic scale. The appearance of the minimum structure marks the disappearance of the detailed structure of the macroscopic behavior. The existence of the submacroscopic scale reminds us of the Kolmogorov scale in turbulence, where the structure of the energy cascade is destroyed by the mechanism of thermal dissipation.

XIII. SUMMARY AND DISCUSSION

In this paper we discussed the collective motion of globally coupled tent maps from the viewpoint of the dynamical properties of the distribution. We defined the Lyapunov exponent for the collective motion from the growth of a disturbance in the distribution $\rho_n(X)$ through projection on the order parameter h_n. The values obtained for the Lyapunov exponent are nontrivial in the sense that they cannot be inferred from the dynamical properties of one tent map. It was found that the collective motion is chaotic whenever it appears, even though it seems to be periodic or quasiperiodic. The chaotic behavior of the collective motion seems to exist inside the minimum structure of the collective motion, where disturbances grow exponentially. This minimum structure persists for any collective motion with chaotic behavior and its size can be estimated from the value of the Lyapunov exponent. These results suggest the reason that the collective motion exhibits low-dimensional behavior despite the actual high dimensionality.

It is necessary to explain how the minimum structure come into existence. The scaling relations for the size δ_c of the minimum structure, Eq. (12), and the Lyapunov exponent λ for the collective motion, Eq. (10), are similar to the forms obtained analytically by Ershov and Potapov [13]. They considered the size of the fluctuations and Lyapunov exponents for globally coupled tent maps. Due to their abstruse analysis, we are not sure if the fluctuations considered there are related to the minimum structure discussed in this paper. We need a more intuitive explanation regarding the occurrence of the chaotic fluctuations and their confinement within the minimum structure.

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